

Sandpiles on a Sierpinski gasket

Frank Daerden, Carlo Vanderzande

Departement Wiskunde Natuurkunde Informatica
Limburgs Universitair Centrum
3590 Diepenbeek, Belgium

February 1, 2008

Abstract

We perform extensive simulations of the sandpile model on a Sierpinski gasket. Critical exponents for waves and avalanches are determined. We extend the existing theory of waves to the present case. This leads to an exact value for the exponent τ_w which describes the distribution of wave sizes: $\tau_w = \ln(9/5)/\ln 3$. Numerically, it is found that the number of waves in an avalanche is proportional to the number of distinct sites toppled in the avalanche. This leads to a conjecture for the exponent τ that determines the distribution of avalanche sizes: $\tau = 1 + \tau_w = \ln(27/5)/\ln 3$. Our predictions are in good agreement with the numerical results.

Keywords: sandpile model, fractals

PACS-classification: 64.60.Ak, 5.40.+j, 5.70.Ln

corresponding author:

Carlo Vanderzande

Departement WNI

Limburgs Universitair Centrum

Universitaire Campus

3590 Diepenbeek

Belgium

telephone:32/11/268216

fax:32/11/268299

e-mail:cvdzande at luc.ac.be

1 Introduction

The sandpile model [1] plays a special role in non-equilibrium statistical mechanics since it is the standard example of a system that reaches a critical stationary state without the necessity of fine-tuning any parameters. This property, which was named self organised criticality, has been suggested to lie at the basis of the widespread occurrence of fractals and $1/f$ -noise in natural systems [2]. Though the sandpile model seems to have little relevance for the description of real sand, it has the advantage that it can be treated with analytical methods. Hence, it plays a role similar to that of the Ising model in the theory of equilibrium critical phenomena.

In the ten years that have passed since its introduction the model has been intensively studied, mostly in two dimensions. Besides extensive numerical work [3], several theoretical developments have been made [4–8]. These have shown that many properties of the model are closely connected to those of spanning trees, which in turn can be described by the $(q \rightarrow 0)$ -limit of the Potts model [5]. These connections have, for the case of two dimensions, led to exact results for the exponents that describe the distribution of sizes of waves and last waves [6, 7]. Moreover, the relation to spanning trees has led Prietzhev and his coworkers [8] to a conjecture (again in $d = 2$) for the exponent governing for the sizes of avalanches, which in a way is the ‘most wanted’ of the sandpile exponents. The conjectured value is in good agreement with numerical estimates. In the present paper, we present results for the sandpile model on the Sierpinski gasket, a well known deterministic fractal. We have two main reasons for doing this. Firstly, we want to extend the theory for waves and the conjecture for avalanches to the case of the Sierpinski gasket and compare them with the results coming from extensive simulations. This can also lead to a better appreciation of the results in two dimensions. Secondly, the development of effective renormalisation group approaches is a major goal in the field of non-equilibrium statistical mechanics. In this respect deterministic fractals can play an important role since often real space renormalisation approaches can be worked out exactly for them. Some real space approaches to sandpile models have been introduced and the results presented here can be useful in testing their validity [9, 10]. We also mention that we are not the first to study the sandpile model on a Sierpinski gasket [11]. A previous study was however purely numerical and was moreover restricted to an investigation of avalanches.

This paper is organised as followed. In section II we introduce the sandpile model, and give a brief overview of the existing theoretical work. In section III, we present our numerical results for waves and avalanches on the Sierpinski gasket. In section IV, our data are compared with exact results which we can obtain for wave exponents. Based on our numerical results, we also give a conjecture for the exponent τ . Finally we present some concluding remarks in section V.

2 Sandpiles, spanning trees and waves

The (Abelian) sandpile model can be defined on any graph, but for definiteness we will introduce it in the contexts of the square lattice and of the Sierpinski gasket. In both cases each vertex of the lattice has four nearest neighbours. To each such vertex i we associate a height variable z_i which can take on any positive integer number. We also introduce a critical height z_c of the height variable which we will take equal to four for all vertices. Furthermore, it is important to stress

that we work on a finite part of the lattice (which we will denote by \mathcal{L}), which contains N vertices. In the case of the Sierpinski gasket, N is trivially related to the number of iterations n used in constructing the fractal.

The sandpile model has two time scales. On a slow time scale we drop sand at a randomly selected site and thereby increase the height variable by one unit: $z_i \rightarrow z_i + 1$. When at a given site, $z_i > z_c$, we say that the site becomes *unstable* and perform a *toppling* of that site which involves the following operations:

$$z_i \rightarrow z_i + \Delta_{i,j} \quad (1)$$

where

$$\begin{aligned} \Delta_{i,i} &= -4 \\ \Delta_{i,j} &= 1 \quad \text{if } i \text{ and } j \text{ are nearest neighbours} \\ \Delta_{i,j} &= 0 \quad \text{otherwise} \end{aligned}$$

The matrix Δ is the discrete Laplacian. Through toppling, neighbouring sites can become unstable, topple themselves, create new unstable sites, and so on. This *avalanche* of topplings proceeds on the second, very fast, time scale, so that no new grains of sand are added before the avalanche is over. Sand can leave the system when a boundary site topples. Note that (see figure 1) in the Sierpinski gasket there are only three sites through which sand can flow out. An avalanche is over when all sites are stable again. We call also the resulting configuration of height variables stable. It is not difficult to see that the order in which unstable sites are toppled doesn't influence the stable configuration which is obtained when the avalanche is over. This Abelian nature of the sandpile model is crucial in the further analysis of its properties [4].

For further reference it also necessary to introduce the matrix G which is the inverse of Δ and which is therefore the lattice Green function of the Laplacian. It was shown by Dhar that the element G_{ij} of the lattice Green function gives the expected number of topplings at site j after a grain of sand was dropped at site i .

The criticality of the model manifests itself in power law distributions which are found for many quantities. For example, let $D(s, N)$ denote the distribution of the number of distinct sites s toppled in an avalanche. It is found numerically that $D(s, N)$ obeys a scaling law

$$D(s, N) \approx s^{-\tau} F(s/N) \quad (2)$$

Extensive simulations of the sandpile model in two dimensions lead to the estimate $\tau = 1.22$ [3]. It is also found numerically that avalanches are compact, i.e. the fractal dimension D of the set of sites toppled in a avalanche is equal to that of the lattice on which the model is studied. In particular, we have for the case of the Sierpinski gasket of figure 1, $D = \ln 3 / \ln 2$.

The number of stable configurations of the model equals z_c^N . However, it has been shown by Dhar [4] that the number of recurrent configurations in the stationary state is exponentially smaller. In fact, there is a one-to-one correspondance between these recurrent configurations and the number S_N of spanning trees on a lattice \mathcal{L}^* which consists of \mathcal{L} extended with an extra site, known as the sink, which represents the boundary [5]. It is known that S_N grows in general as $\mu^{N+o(N)}$, where μ

is called the *connective constant* of spanning trees [12]. Spanning trees are described by the q -state Potts model in the limit $q \rightarrow 0$ [13]. The same model also describes resistor networks.

Because of the Abelian nature of the sandpile model, we can perform the toppling of unstable sites in any desired order. A particularly interesting order leads to the concept of *waves* [7]. Suppose that an avalanche starts at a site i_0 and that after some topplings i_0 becomes unstable again. One can then keep i_0 fixed for the moment and continue with the toppling of other sites until all of them (apart from i_0) are stable. It is easy to show that in such a sequence all sites topple at most once. This set of topplings is called *the first wave*. Next, we topple the site i_0 for the second time. If after some topplings it becomes unstable again, we keep it fixed, and topple all the other unstable sites. This set of topplings constitutes the second wave. We continue in this way until we finally reach a stable configuration. We can in this way decompose any avalanche in a set of waves. Of course, subsequent waves in an avalanche are highly correlated, but we can also consider the statistical properties of the set of all waves, irrespective of the avalanche in which they occur. In this way, we can look at the distribution $\hat{D}(s_w, N)$ of the number of sites s_w toppled in any wave. Again, it is found that $\hat{D}(s_w, N)$ has a scaling form

$$\hat{D}(s_w, N) \approx s^{-\tau_w} \hat{F}(s_w/N) \quad (3)$$

where we have introduced the exponent τ_w for waves.

The properties of the last wave in a given avalanche are also of interest. The distribution of the number of sites s_{lw} toppled in this last wave leads to the introduction of still another exponent, denoted as τ_{lw} . Because $s_{lw} \leq s$, we arrive at the obvious upper bound

$$\tau \leq \tau_{lw} \quad (4)$$

It turns out that the properties of waves are easier to analyse than those of the avalanches. Without going into details we summarize the following important results:

- There is a one-to-one correspondence between waves and two-rooted spanning trees (one root corresponds with the sink, the other with the site i_0) on \mathcal{L}^* [7]. Furthermore, the element G_{ij} of the lattice Green function is given by the ratio of the number of two rooted spanning trees (in which i and j are in the same subtree) to the number of (one rooted) spanning trees S_N . Hence, properties of waves can be determined from G . From the known asymptotic behaviour of the lattice Green function on an Euclidean lattice, $G_{ij} \sim |i - j|^{2-D}$, one then finds

$$\tau_w = \frac{2D - 2}{D} \quad (5)$$

- The exponent τ_{lw} can be obtained from the probability that b sites get disconnected from a spanning tree when a randomly chosen bond of the tree is deleted. In particular, one finds that τ_{lw} can be expressed in terms of the chemical dimension z of spanning trees [6]. On an Euclidean lattice the precise result is

$$\tau_{lw} = 1 + \frac{2 - z}{D} \quad (6)$$

From these results, one obtains, in $D = 2$ (where $z = 5/4$ [5])

$$\tau_w = 1 \qquad \tau_{lw} = 11/8$$

A final theoretical development which we need to discuss leads to a conjecture on the exponent τ [8]. Priezzhev *et al.* first relate τ to another exponent, α . To introduce this exponent, it is assumed that the number of sites toppled in consecutive waves (say the k -th and $k + 1$ -th wave) decreases. The change in size, $\Delta s_w = s_w(k) - s_w(k + 1)$ is assumed to occur in a self-similar way, so that

$$\Delta s_w \sim s_w^\alpha \tag{7}$$

One can then immediately show [8] that

$$\tau + \alpha = 2 \tag{8}$$

Finally, using the relation between waves and two-rooted spanning trees, and considering possible processes by which the size of a wave can decrease, it is found that $\alpha = 3/4$ and hence $\tau = 5/4$. This prediction is in excellent agreement with numerical estimates, and may very well be exact.

3 Numerical results

In order to obtain numerical estimates for the exponents τ , τ_w , τ_{lw} and α on the Sierpinski gasket we have performed extensive simulations of the sandpile model on this fractal structure. In these simulations, we have considered lattices with $n = 3$ up to $n = 8$ (the latter contains $N = 9843$ sites). For each of these we have studied more than $(1000\ N)$ avalanches.

In figure 2, we show our results for the distribution of the size of the waves. As can be seen, log-periodic oscillations are superposed on the power law decay of the distribution. The behaviour of $D(s, N)$ can therefore be described by complex critical exponents. This is a well known phenomenon in systems with *discrete* scale invariance [14]. The imaginary part of the critical exponent is in a sense trivial since it can be related to the discrete rescaling factor of the fractal lattice. From a careful analysis of the data one can hence obtain finite size estimates of the exponent $\tau_w(n)$ as a function of n . These can be extrapolated by making the usual assumption

$$\tau_w(n) = \tau_w + \frac{C}{\ln n} \tag{9}$$

In figure 3 we show this extrapolation. It leads to the estimate

$$\tau_w = 0.47 \pm 0.05 \tag{10}$$

It is difficult to make a precise determination of the error in this estimate. The error we give is therefore more indicative.

Figure 4 shows our results for the distribution of the size of the last wave. From an examination of these data, we find

$$\tau_{lw} = 1.66 \pm 0.05 \tag{11}$$

Finally, our data for the distribution of avalanche sizes, shown in figure 5, yield the exponent estimate

$$\tau = 1.46 \pm 0.05 \quad (12)$$

in agreement with the result found by Kutnjak-Urbanc *et al.* [11], who obtained $\tau = 1.51 \pm 0.04$.

We have also investigated the behaviour of Δs_w in order to determine the exponent α . To obtain this exponent, we have to consider only those cases in which $\Delta s_w > 0$ (in fact, there are many cases for which the opposite inequality holds). In figure 6 we show our data (after integration over bins). As can be seen from the figure, it is not easy to determine a clear value of α . Our best estimate is

$$\alpha = -0.15 \pm 0.1 \quad (13)$$

In the next section, we will give a lower bound for α . Our estimate (13) is very close to that bound. It is however difficult to imagine how α can be negative. We therefore turned to an alternative way to determine α . This is done by investigating how the average number n_w of waves in an avalanche scales with the size of the avalanche. When the size of consecutive waves decreases, the size of the first wave is proportional to the size of the whole avalanche. From (7) we then obtain

$$n_w \sim s^{1-\alpha} \quad (14)$$

In figure 7 we show data for this quantity for $n = 6$ and $n = 8$.

We find *very clear evidence for a linear relation between n_w and s* . This relation breaks down only for the largest avalanches, which are those that span the lattice. But, as pointed out by Kutnjak-Urbanc *et al.* [11], these avalanches are somewhat peculiar. They occur here because only through such avalanches is it possible for sandgrains to leave the system.

From the linear relation we find

$$\alpha = 0.0 \pm 0.01 \quad (15)$$

These data lead us to conjecture that $\alpha = 0$ exactly. As we will see in the next section, this result will be very useful in determining the exponent τ .

4 Exact and conjectured results

We now present some exact results for the connective constant μ for spanning trees (and hence for the number of recurrent states) and for the exponents τ_w and τ_{lw} . We make also a conjecture for τ .

As discussed in section 2, there exists a close connection between the theory of the sandpile model and the ($q \rightarrow 0$)-limit of the Potts model. This connection should allow the determination of several properties of the sandpile from a study of that model on the Sierpinski gasket. For example, the connective constant of the spanning trees can be obtained from the free energy of the Potts model when $q \rightarrow 0$. This free energy in turn can most easily be obtained by performing a simple but

exact real space RG calculation for the Potts model on the Sierpinski gasket. Consider therefore the q -state Potts model for which we will denote the spin variable at vertex i by σ_i . On a Sierpinski gasket, these spins can also be labeled according to the ‘generation’ in which they appear during the iterative construction of the fractal. The reduced Hamiltonian $H(K)$ of the Potts model is given by

$$H(K) = K \sum_{\langle i,j \rangle} \delta_{\sigma_i, \sigma_j} \quad (16)$$

A renormalisation group for $H(K)$ on the Sierpinski gasket can be obtained by performing a partial trace, denoted by Tr^* over the spin variables which have the same ‘generation label’:

$$e^{H(K')} + g(K) = \text{Tr}^* e^{H(K)} \quad (17)$$

The free energy of the model can then be obtained by a summation of $g(K)$ along the renormalisation group trajectory. We have performed such a calculation. After sending $q \rightarrow 0$ the RG flow is remarkably simple

$$K' = \frac{3}{5}K \quad (18)$$

$$e^{g(K)} = 50K^3 q^{3/2} \quad (19)$$

from which we obtain the connective constant for spanning trees after a straightforward calculation. The result is

$$\mu = 50^{1/3} \sqrt{3/5} \approx 2.854 \quad (20)$$

As mentionned in section 2, the exponent τ_{lw} can be obtained from the probability $P_s(b)$ that upon deleting a bond at random from a spanning tree, a subtree of size b gets disconnected. On an Euclidean lattice this relation leads to the result (6). On a fractal lattice, the relation has to be modified. The correct expression now also involves the fractal dimension D_w of a random walk on the fractal. This was shown in reference [15], where it was found that

$$P_s(b) \sim b^{-D_w/z}$$

Then, repeating the arguments in [6] we obtain

$$\tau_{lw} = 1 + \frac{D_w - z}{D} \quad (21)$$

This is a quite general result for τ_{lw} which should for all cases (fractal and Euclidean).

For spanning trees on the Sierpinski gasket, z was recently determined by Dhar and Dhar in a study of loop-erased random walks [15]. Their result is

$$z = \frac{\ln [(20 + \sqrt{205})/15]}{\ln 2} \quad (22)$$

[Before proceeding, we would like to point out a general recipe for determining z , using again an RG approach. For a tree, the chemical dimension equals the red bond dimension D_R . The red

bonds of a connected graph are those edges which when cut, disconnect the graph. Coniglio [16], and independently Saleur and Duplantier [17], pointed out that the red bond dimension for the critical Potts model can be obtained by adding a term

$$M \sum_{i,j} \delta_{\sigma_i,1} \delta_{\sigma_j,1}$$

to the Potts hamiltonian (16). The red bond dimension can then be obtained from the linearised RG equation for M at the critical Potts fixed point. Hence, it is possible to get z for spanning trees by performing an RG calculation (either exact or approximate) on an extended Potts model. We will present results from such an approach in a future publication.]

Inserting (22) in (21) and using the value of D_w for the Sierpinski gasket ($D_w = \ln 5 / \ln 2$, [18]) we get

$$\tau_{lw} = 1 - \frac{\ln[(20 + \sqrt{205})/75]}{\ln 3} \approx 1.712 \quad (23)$$

This prediction is in good agreement with the numerical value found in section 3.

Next, we turn to the exponent τ_w . This should be determined from the behaviour of the Green function. We will assume that the Green function G_{ij} decays for large $r = |i - j|$ as

$$G_{ij} \sim r^{-2x} \quad (24)$$

The Green function gives the expected number of topplings at site j when a grain of sand has been dropped at site i . Hence it consists of all waves that cover the distance between site i and j . Since waves are compact, we obtain

$$G_{ij} \sim \int_{r^D}^{\infty} \hat{D}(y) dy$$

where we have neglected finite size effects. From (24) we then get a general expression for the exponent τ_w

$$\tau_w = 1 + \frac{2x}{D} \quad (25)$$

To determine τ_w for the particular case of the Sierpinski gasket, we only need to determine the value of x . This exponent was determined in early studies on the behaviour of the resistance on fractals with the result [19]

$$2x = \frac{\ln(5/3)}{\ln 2}$$

This leads to the prediction

$$\tau_w = \frac{\ln(9/5)}{\ln 3} \approx 0.535 \quad (26)$$

This value is again in agreement with the one determined numerically. In principle, the exponent x can also be obtained from the $q \rightarrow 0$ limit of the Potts model. In fact, it can be obtained from the renormalisation group equation for K , see (18).

Finally, we turn to the exponents τ and α . It is easy to generalise the result (8) which was obtained in $D = 2$. The size s_w of any wave must be less than or equal to the size s of the avalanche in which it occurs. Moreover, the number of waves whose size lies between s_w and $s_w + ds_w$ is according to (7) proportional to $s_w^{-\alpha}$. Combining these two results we obtain for the size distribution of waves

$$\hat{D}(s_w) \sim s_w^{-\alpha-\tau-1}$$

from which we immediately obtain the appropriate generalisation of (8)

$$\alpha + \tau = 1 + \tau_w \quad (27)$$

Our numerical results (10), (12) and (15) are nicely consistent with this picture. From the upper bound (4), we obtain a bound on α

$$\alpha > 1 + \tau_w - \tau_{lw} \approx -.177$$

Our first estimate of α , given in (13), is just consistent with this bound.

Most interestingly our conjecture that $\alpha = 0$ leads to a conjecture for τ

$$\tau = 1 + \tau_w = \frac{\ln(27/5)}{\ln(3)} \approx 1.535 \quad (28)$$

which is in excellent agreement with our numerical results.

We have also tried to extend the argument of Priezzhev and his coworkers for the exponent α to the fractal case. As far as we could see, such an argument always leads to a value of the exponent α which is of order one, and which therefore is completely in disagreement with the numerical result. It therefore seems that the way in which consecutive waves are related is different for the fractal case than for the Euclidean one.

5 Discussion and conclusions

In this paper, we have presented extensive simulations for waves and avalanches on the Sierpinski gasket. One of the most surprising results is the fact that the exponent α is very close to zero. This implies that the way in which waves become smaller is almost independent of their size. This is most probably a direct consequence of the fractal nature of the Sierpinski lattice.

We have also extended the existing theory for waves to the present case. This has led to exact expressions for the exponents τ_{lw} and τ_w which are close to the numerically determined values.

Most interestingly, assuming that α is zero, we are led to the conjecture (28) for the exponent τ . Also this prediction is in good agreement with our simulation results.

In this way, there is now a second situation (after the two-dimensional case) for which wave exponents are determined exactly and for which a conjecture for the τ -exponent exists.

The result $\alpha = 0$ indicates, in our opinion, that there is good hope that the sandpile model on a Sierpinski gasket may be solved exactly. We are currently using real space RG approaches which have been introduced for sandpile models [9, 10], to see whether they may be worked out exactly on this lattice.

Acknowledgement One of us (CV) would like to thank D. Dhar and V. Priezzhev for useful discussions on the subject of this paper. We also thank the Inter University Attraction Poles for financial support.

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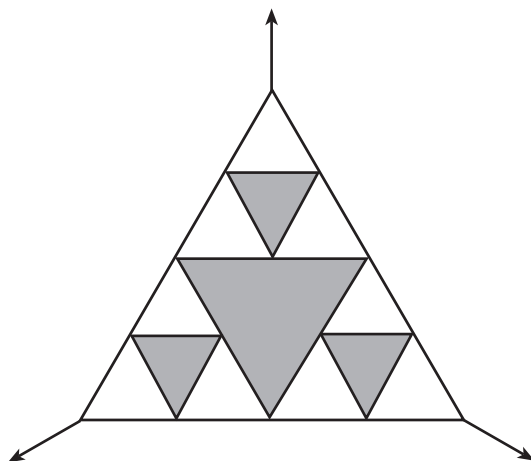


Figure 1: The simple Sierpinski gasket with $n = 2$. Sand can only leave the lattice through the three sites indicated by an arrow.

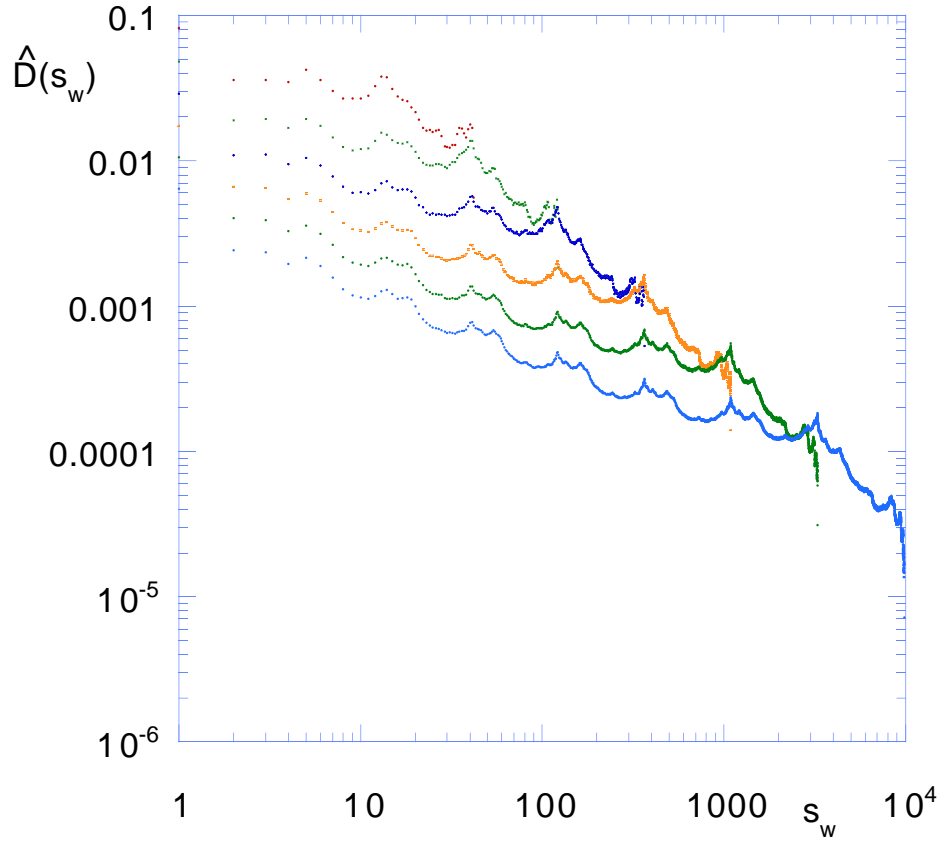


Figure 2: Distribution of wave sizes. The data give numerical results for $n = 3$ to $n = 8$ (top to bottom).

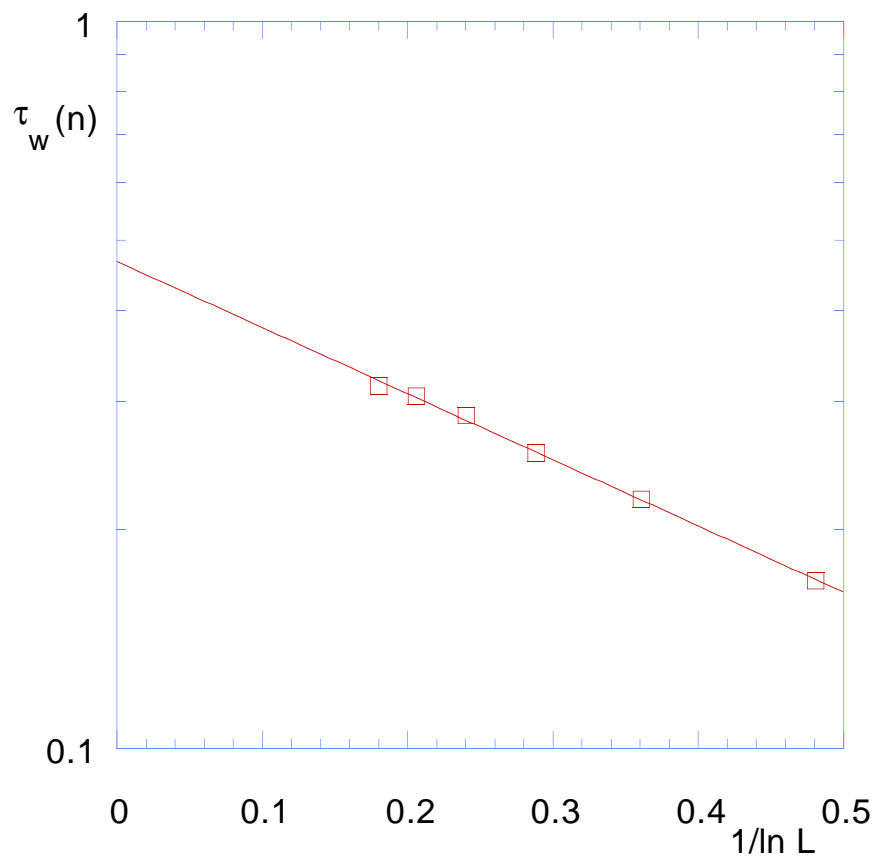


Figure 3: Extrapolation of $\tau_w(n)$ ($L = 2^n$).

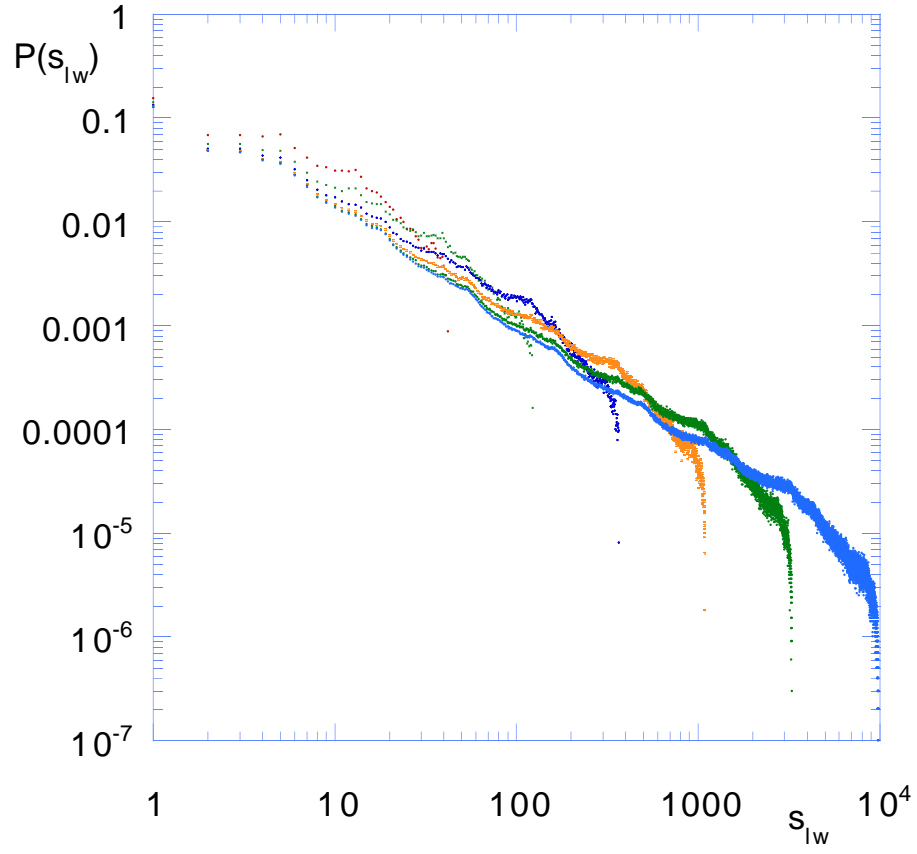


Figure 4: Distribution of last wave sizes. Data are given for $n = 3$ to $n = 8$.

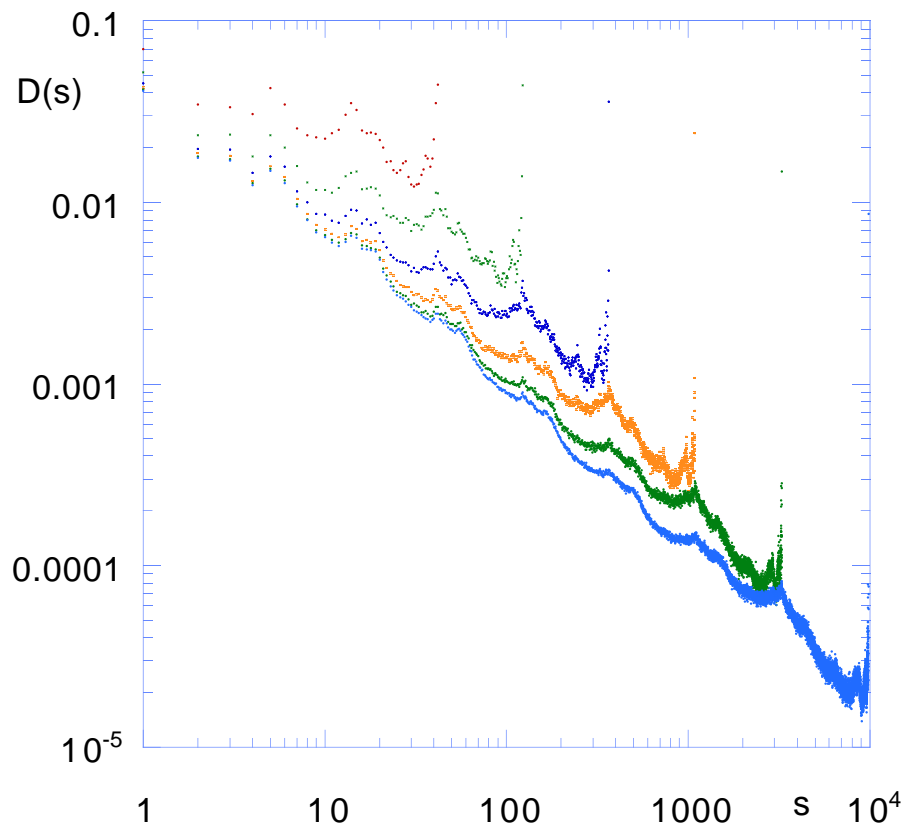


Figure 5: Distribution of avalanche sizes. Data are given for $n = 3$ to $n = 8$.

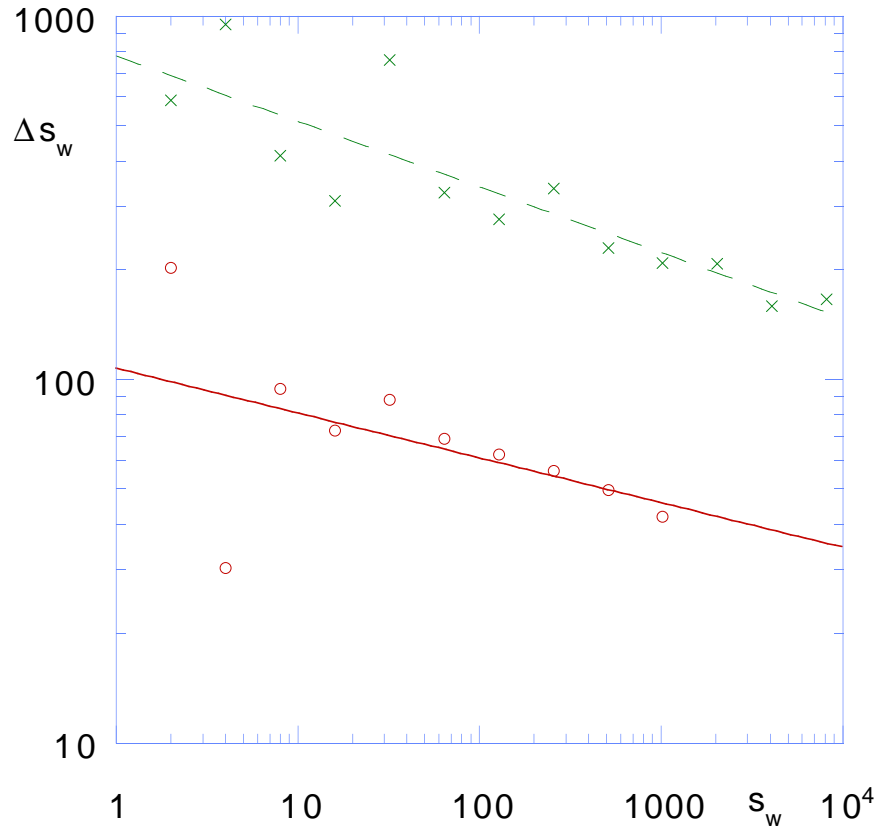


Figure 6: Decrease in size (averaged over bins) of consecutive waves, versus wave size, for $n = 6$ and $n = 8$ (upper dataset).

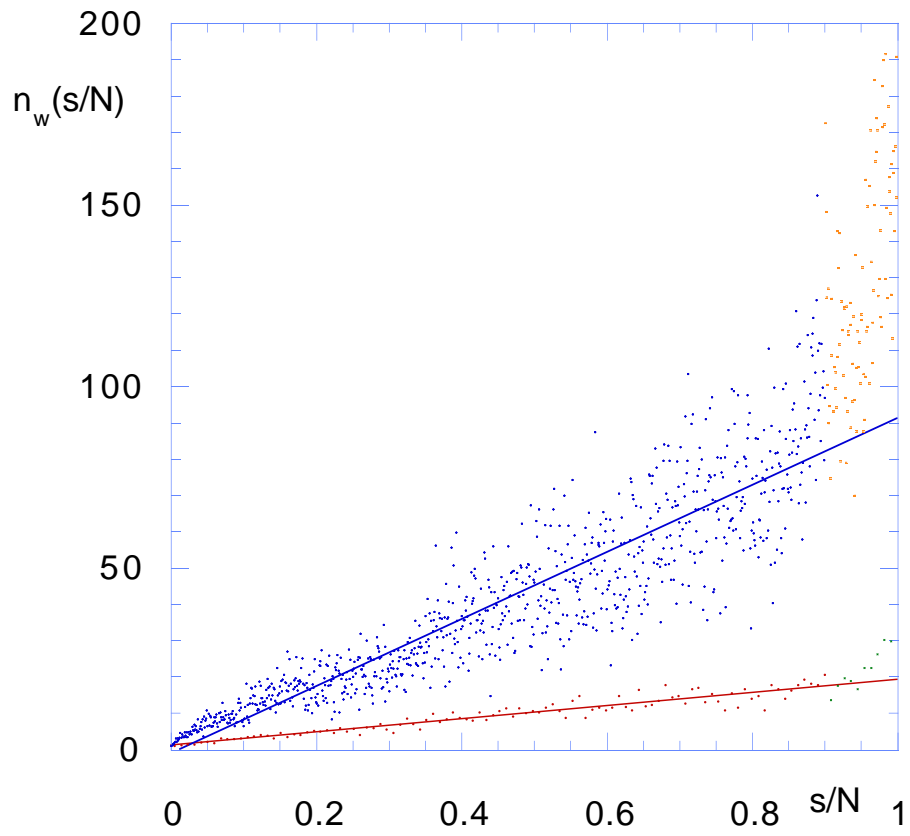


Figure 7: Number of waves in an avalanche versus size of the wave, for $n = 6$ and $n = 8$ (upper dataset).